Indian Statistical Institute
Mid-Semestral Examination 2007-2008
B.Math (Hons.) II Year
Analysis III

Time: 3 hrs | Date:17/09/2007 | Marks : 49
Neatness : 1
Total : 50

1. Let $g_1,\ g_2 : [a,b] \to R$ be continuous functions such that $g_1(x) \le g_2(x)$ for all x. Let $f : R^2 \to R$ be any bounded, uniformly continuous function. Define $h : [a,b] \to R$ by
$$h(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)\ dy.$$
Show that h is a continuous function. [6]

2. Let X be any nonempty set. Let $f : X \to R$ be any bounded function. Show that $\sup\limits_{x,y \in X} |f(x) - f(y)| = \sup\limits_{x,y \in X} [f(x) - f(y)] = \sup\limits_{x \in X} f(x) - \inf\limits_{y \in Y} f(y)$. Any of these numbers is called oscillation of f on the set X and is denoted by $\mathrm{Osc}(f, X)$. [3]

3. Let $f : [0,1] \to R$ be any bounded function.
(a) Show that f has a limit at $0 \Leftrightarrow$
$$\lim_{\delta \to 0} \mathrm{Osc}(f, (0,\delta)) = 0.$$
[2]
(b) f is continuous at $0 \Leftrightarrow$
$$\lim_{\delta \to 0} \mathrm{Osc}\ (f, [0,\delta]) = 0.$$
[2]

4. Let $\alpha : R \to R$ be given by
$$\begin{aligned} \alpha(x) &= 0 \text{ for } x < x_0 \\ &= 1 \text{ for } x > x_0 \\ \alpha(x_0) &= \frac{1}{2}. \end{aligned}$$
Let $a < x_0 < b$. Let $f : [a,b] \to R$ be any bounded function and f be Riemann-Stieltjes integrable w.r.t. α. Show that f is continuous at x_0. [Hint: You can use (3)(b) and a similar result if necessary.] [4]

5. Let $h_1, h_2 : [c, d] \to R$ be continuous functions such that $h_1(y) \le h_2(y)$. Let

$$G = \{(x, y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}.$$

Let $Q : R^2 \to R$ be a continuous function such that $\frac{\partial Q}{\partial x}$ exists and is continuous. Let Γ be the boundary of G traversed in the anti clockwise direction. Verify that

$$\int\int_G \frac{\partial Q}{\partial x}(x, y)\ dx\ dy = \int_\Gamma Q\ dy.$$

[4]

6. Let $u, v : R^2 \backslash (0, 0) \to R$ be given by

$$u(x, y) = \frac{x}{x^2 + y^2} \qquad v(x, y) = \frac{-y}{x^2 + y^2}.$$

(a) Calculate $\int_\Gamma v\ dx + u\ dy$ where Γ is the boundary of any rectangle $[a, b] \times [c, d] \subseteq R^2 \backslash (0, 0)$. [If necessary you can use Greens theorem].

[2]

(b) Show that

$$\int_{x^2+y^2=1} v\ dx + u\ dy = 2\pi.$$

[2]

(c) Why we cannot find $f : R^2 \backslash (0, 0) \to R$ such that

$$\frac{\partial f}{\partial x} = v \text{ and } \frac{\partial f}{\partial y} = u?$$

[2]

7. (a) Let (X, d) be a metric space. $f_1, f_2, \ldots$ be a sequence of real valued bounded uniformly continuous functions. If $g : X \to R$ is bounded and $\lim_{n\to\infty} \sup_{x \in X} |g(x) - f_n(x)| = 0$. Show that g is uniformly continuous. [3]

(b) Let $g : (0, 1] \to R$ be given by $g(x) = \sin(\frac{1}{x})$. Show that nobody can find a sequence $P_1, P_2, \ldots$ of polynomials such that $\lim_{n\to\infty} \sup_{x \in (0,1]} |P_n(x) - g(x)| = 0$. [2]

8. Let $f_1, f_2, \ldots : [0, 1] \to R$ be C^1 functions and $h, g : [0, 1] \to R$ be continuous functions such that $f_n \to g$ uniformly and $f_n' \to h$ uniformly on $[0, 1]$. Show that g is C^1. [3]

9. Let (X, d) be a metric space

(a) $\zeta = \{f : X \to R$ continuous and bounded$\}$ with the metric D on ζ given by $D(f, g) = \sup\limits_{x \in X} |f(x) - g(x)|$. Show that (ζ, D) is a complete metric space. [7]

(b) Let (Y, m) be a metric space. Let $g_1, g_2, \ldots : Y \to R$ be bounded and continuous. Let $M_k = \sup\ |g_k(x)|$. Assume that $\sum M_k < \infty$. Define $g(x) = \sum\limits_k g_k(x)$. Show that g is a continuous function. [3]

10. Let $f : [a, b] \to R$ be a continuous function

(a) Let $\int\limits_a^b f(x)\ x^n\ dx = 0$ for $n = 0, 1, 2, 3, \ldots$. Show that $f \equiv 0$. [3]

(b) Let $\int\limits_a^b f(x)\ (x - k)^n = 0$ for a fixed real $k \not\equiv 0$ and $n = 0, 1, 2, 3, \ldots$. Show that $f \equiv 0$. [1]